GENERAL REDUCIBILITY AND SOLVABILITY OF POLYNOMIAL EQUATIONS

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ABSTRACT
A complete work on general reducibility and solvability of polynomial equations by algebraic means-radicals is developed. These equations called, reanegbèd and vic-emmeous are designed by using simple algebraic principles on how systems of equations and polynomials behave. Reanegbèd equations are capable of reducing or rewriting all polynomial equations to desired forms while vic-emmeous equations are capable of extracting out a root or roots from two or more algebraically closed polynomials. Unlike quadratic, cubic, and quartic polynomials, the general quintic and higher degree polynomials cannot be solved algebraically in terms of finite number of additions, subtractions, multiplications, divisions, and root extractions as rigorously demonstrated by Abel (1802 –1829) and Galois (1811 –1832). However, allowing the use of reanegbèd equations and vic-emmeous equations make reducing and solving all polynomial equations possible algebraically in terms of finite number of additions, subtractions, multiplications, divisions, and root extractions.

Keywords: Roots of polynomial equations, Polynomials (irreducibility, etc.), Galois Theory, Solving Polynomial Systems, Polynomial factorization, Polynomial Ring

MSC2010 Subject Classification. Primary, 65H04; Secondary, 11R09, 11520, 13P15, 13P05, 13F20

INTRODUCTION
A polynomial equation in variable $x$ is an equation of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

in which the coefficients $a_0, a_1, a_2, \ldots, a_n$ can be any number, with the exception that $a_0 \neq 0$. The degree of the polynomial equation is equal to the positive integer $n$. A root is a value of $x$ that when plugged into the polynomial equation yields 0; a polynomial equation is solved when all the roots of the equation have been found. Fundamental theorem of algebra gives way to every non-zero, single-variable polynomial of degree $n$ with complex coefficients to have exactly $n$ roots if we count multiplicity. Coefficients of a polynomial with real numbers are considered as complex with zero imaginary numbers.

A linear equation is an equation of the first degree and has only one root. The single root of the linear equation $ax + b = 0$ is $x = -\frac{b}{a}$

The quadratic, or second-degree, equation, $ax^2 + bx + c = 0$ has two roots, given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solutions of cubic and quartic equations were also given by Gerolamo Cardano (1545) and Lodovico Ferrari (1522 –1565) respectively. Solutions of degree five and above equations were tackled seriously by great mathematicians many years ago. At the beginning of the 19th century, Galois (1811 –1832) and Abel (1802 –1829) demonstrated serious efforts towards answering the general algebraic reducibility and solvability of polynomial equations to the extent of giving out condition necessary for an equation to be solvable.
or not. They all proved and concluded to be impossible generally.

This proposed general solution for all quintic equations would not only contradict Abel-Ruffini, but also would indicate that there are severe problems with the correctness of all of Galois Theory.

In this paper, we shall develop a new method for solving all polynomial equations but much attention will be on all quintic equations including the type since similar steps can be applied to degree six and above.

**Definition 1.1** Let \( f(x) \) denote a polynomial. We say \( f(x) \)
- is just if the terms in \( f(x) \) are written in a descending order of powers otherwise \( f(x) \) is unjust.
- has dipole terms if \( f(x) \) has terms in the lowest and highest powers. The highest power's term is the leading dipole term and the lowest power's term is the following dipole term.
- has an intermediate term if there exist at least a term apart from its dipole.

**Example:** If the polynomial \( q + mx^3 + nx^2 + px \) is arranged in descending order of powers such as \( mx^3 + nx^2 + px + q \) that the powers are starting from 3 to 2 to 1 and to 0, then we refer to the polynomial \( mx^3 + nx^2 + px + q \) as Just. The two terms \( mx^3 \) and \( q \) of the Just polynomial are referred to as dipole while \( nx^2 \) and \( px \) become the intermediate terms.

**Remark**
Terminologies discussed in definition 1.1 will be helpful in terms of description and analysis in the subsequent work.

**Reducibility of polynomial equations**
Reducing polynomial equations in this paper means transforming an equation to its simplest form or rewriting the equation to a desired (standard) form. A typical example of equations which are being re-written after going through transformation is \( x^5 + ax + b = 0 \). Equations of the type \( x^5 + ax + b = 0 \) will first be transformed to an equivalent equation of \( x_1^5 + \alpha x_1^4 + \beta x_1^3 + \gamma x_1^2 + \delta = 0 \) before transforming again to the standard equation \( x^5 + a_1 x_1^4 + b_1 x_1^3 + c_1 x_1^2 + d_1 = 0 \). Solutions of \( x_1^5 + a_1 x_1^4 + b_1 x_1^3 + c_1 x_1^2 + d_1 = 0 \) are discussed in proposition 3.15 but before that let us show how we reduce all polynomial equations.

**Definition 2.1.** Let \( f(x) \) be a polynomial with degree \( k \); \( f_{++}(x) \) denotes a proper polynomial if
- \( f(x) \) has at least one intermediate term in the expression and
- \( k \geq 2 \).

Otherwise \( f(x) = f_+(x) \) is called Improper polynomial and typical examples are degree one and cyclotomic polynomials without any intermediate term but dipole terms.

**Definition 2.2.** A reanegbèd equation is an auxiliary equation which helps in reducing or rewriting a proper polynomial equation by eliminating a particular intermediate term.

**Definition 2.3.** Negation of root \( \mu \) is the product of the derivative of the proper polynomial at \( \mu \) and the square of its imaginary unit \( (i) \). It is denoted by the letter \( t \) and the equation formed is called reanegbèd equation.

For \( f_{++}(x) \) being proper, the negation of root \( \mu \) is given by
\[
 t_n = i^2, f_{++}^n(\mu) \text{ for } \mu \in f_{++}(x) = 0, \text{ so that } t_n = (-1)^2, f_{++}^n(\mu) \text{ or }
\]
\[
(2.4) \quad t_n + f_{++}^n(\mu) = 0 \text{ for } \mu \in f_{++}(x) = 0
\]
Equations of (2.4) are called reanegbèd of \( f_{++}(x) \).

**Proposition 2.5.** A polynomial equation of degree \( n = 1 \) is not a proper polynomial equation and therefore has no reanegbèd equation.

**Proof.** By definition 2.1, polynomials with degrees \( n \geq 2 \) are proper polynomials. If \( n < 2 \), then the polynomial becomes improper (i.e. degree one polynomial). For \( n = 1 \), the degree of the first derivative will be \( n - 1 = 1 - 1 = 0 \) which implies no term of variable. No term of variable implies no reanegbèd equation and that all proper
polynomials have reanegbèd equations by (2.4) and definition 2.3 ■

**Proposition 2.6.** For every root $\mu$ of a proper polynomial equation $f_{++}(x)$, there exist (negation of the root) and derivative of $f_{++}(x)$ at $\mu$ making up a reanegbèd equation.

**Proof.** Refer to (2.4) and definition 2.3 for a reanegbèd equation. In particular, the only reanegbèd equation of $f_{++}(x) = x^2 + 5x + 6 = 0$ will be $t_1 + f_1'_{++}(x) = t_1 + 2\mu + 5 = 0$ by proposition 2.5 with roots of $f_{++}(x) = 0$ being $-2$ and $-3$. The root $-2$ will have $t = -1$ and $-3$ will also have $t = +1$. Each root has its corresponding $t$ which satisfies the reanegbèd equation. ■

**Proposition 2.7.** All improper polynomial equations are solvable by algebraic means.

**Proof:** by definition 2.1, all improper polynomial equations are degree one and some cyclotomic equations like $x^n - 1 = 0$ where $n$ is a positive integer. Degree one equations are linear equations and such equations are solvable by algebraic means. Solutions for cyclotomic equations are also given by $e^{2\pi ik/n}$ where $0 \leq k < n$. ■

**Proposition 2.8.**
The degree of a proper polynomial equation is always greater than any of the degrees of its reanegbèd equations.

**Proof.** Proper polynomial $f_{++}(x)$ with degree $n$ will have degrees of $n+1$ as first derivative $f_1'_{++}(x)$, $n-2$ as second derivative $f_2'_{++}(x)$ until we get to $n = 1$. Since derivative of a proper polynomial at a root and its negation constitute the formulation of a reanegbèd equation and all the powers of the derivatives are $< n$, that is $n > n - 1 > n - 2 > 1$. This implies degree of proper polynomial $> degree of its reanegbèd equations. ■

**Proposition 2.9.** The number of reanegbèd equations present in a proper polynomial equation is equal to the degree of the proper polynomial equation minus one i.e. $n - 1$ where $n$ is the degree of proper polynomial equation.

**Proof.** By definition 2.1, polynomials with degrees $n \geq 2$ are proper polynomials. If $n < 2$, then the polynomial becomes improper (i.e. degree one polynomial). In particular, we take $n = 2$ and the degree of the first derivative will be $n - 1 = 2 - 1 = 1$. If $n = 1$ then by proposition 2.5, the second derivative will turn to zero which means reanegbèd equation cannot be formulated at this stage. The only reanegbèd equation present will be from that of the first derivative. Hence one reanegbèd equation present for $n = 2$ satisfying $nR = n - 1$. Similarly for $n = 3, n = 4$ and above. ■

**Proposition 2.10.** The power of a term that will be eliminated in a reduced equation is equal to the difference between the degrees of the proper polynomial and any of its reanegbèd equations i.e. $E = P - R$

**Proof.** In particular, the general cubic equation $x^3 + bx^2 + cx + d = 0$ will have two reanegbèd equations present by proposition 2.9 and they are $3x^2 + 2bx + c + t_1 = 0$ and $6x + 2b + t_2 = 0$. Let us choose the reanegbèd equation

(2.12) $3x^2 + 2bx + c + t_1 = 0$

By proposition 3.5, we can solve (2.11) and (2.12) vic-emmeously to get

(2.13) $x = \frac{b(c+t_1)-9d}{6c-2b^2-3t_1}$ where $3t_1 \neq 6c-2b^2$

From proposition 3.7, (2.13) is a crack. Substituting (2.13) into (2.12) or (2.11) will give a reduced equation,

(2.14) $t_1^3 + pt_1^2 + q = 0$

Where $p = c - \frac{b^2+2(6c-2b^3)}{3}$ and $q = \frac{3(bc-9d)^2 + 2b(bc-9d)(6c-2b^3) + c(6c-2b^3)^2}{9}$

From (2.14) the eliminated term’s power (linear term) satisfies the formula $E = P - R$ i.e. $1 = 3 -$
2. Similarly to the other reanegbèd equation \( 6x + 2b + t_2 = 0 \) which also obey the formula \( E = P - R \) i.e. \( 2 = 3 - 1 \) by eliminating the quadratic term to get a reduced equation \( t_2^3 + pt_2 + q = 0 \).

**Remark:** Proposition 2.10 makes it clear to us that in the process of transformation, each reanegbèd equation of a proper polynomial determines the type of reduced equation to have at the end. This tells us that a general quadratic equation can be proved by this proposition if we eliminate the linear term.

**Example:**

Let's use **proposition 2.10** to prove the quadratic formula, \( ax^2 + bx + c = 0 \) as the general equation of quadratic.

First we can see clearly that \( ax^2 + bx + c = 0 \) is a proper polynomial equation. The only reanegbèd equation of the proper polynomial equation is \( 2ax + b + t = 0 \). Let
\[
(\bigtriangledown) \quad ax^2 + bx + c = 0 \quad \text{and} \quad (\sigma) \quad 2ax + b + t = 0
\]

We solve \((\bigtriangledown)\) and \((\sigma)\) vic–emmeously to get
\[ bx + 2c - tx = 0 \quad \text{and therefore} \]
\[
(\bigtriangledown) \quad x = \frac{2c}{t-b}
\]

We substitute \((\bigtriangledown)\) into either \((\bigtriangledown)\) or \((\sigma)\)

It implies \( 2a \left( \frac{2c}{t-b} \right) + b + t = 0 \) or
\[
4ac + t^2 - b^2 = 0
\]
\[
\therefore \quad t = \pm \sqrt{b^2 - 4ac} \quad \text{and substituting it into equation} \quad (\sigma) \quad \text{will give} \]
\[
2ax + b \pm \sqrt{b^2 - 4ac} = 0
\]
\[
\text{Hence} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

**Solvability of polynomial equations:**

Many attempts have been made on general solvability of polynomial equations and the result still remains impossible to the world before the publication of this paper. This was due to the fact that the technique we used in solving proper polynomial equations of degree two to four was different from that of degree five and above.

Solutions of degree five and above polynomial equations were considered as similar as degree two to four and when such presumption is taken, it will always lead to the proof of the impossibility theorem by Niels Henrik Abel (1802-1829) and Paolo Ruffini (1735-1822) and finally Evariste Galois’ (1811-1832) conclusion in all of his theory.

To give birth to possibilities, we shall introduce the theory (E-Theory) that explains why J-sac families (degree two to four polynomials ) are different from that of B-sac families (degree five and above) and how families of polynomials give solutions by algebraic means- in terms of finite number of additions, subtractions, multiplications, divisions, and root extractions.

**Definition 3.1.**

A family \( F \) is a group of polynomials with the same characteristics. We say a polynomial \( g(x) \in F \) if and only if \( g(x) \) has leading and following dipole terms.

If \( g(x) \in F \) and \( g_r(x) = 0 \) is the reduced equation of \( g(x) \), then we say:

(a) \( g(x) \in J\text{-sac} \) if and only if the reduced equation \( g_r(t) = 0 \) has at most two intermediate terms.

(b) \( g(x) \in B\text{-sac} \) if and only if the reduced equation \( g_r(t) = 0 \) has at least three intermediate terms.

(c) \( g(x) \in \text{Irreducible} \) if and only if \( g(x) \) has no intermediate term and cannot be reduced.

**Proposition 3.2.** All proper polynomial equations of degree \( \leq 4 \) belong to J-sac family.

**Proof.** In particular, let \( h(x) = x^4 + ax^3 + bx^2 + cx + d = 0 \) where the degree of polynomial \( k = 4 \) with three intermediate terms \( x^3, bx^2 \) and \( cx \). The polynomial \( h(x) \) is proper and thus \( h(x) \in F \). We need to investigate the type of family \( h(x) \) belongs. To do that, we check whether the reduced equation \( h_r(t) = 0 \) has at least an intermediate term. \( h_r(t) = 0 \) will have two intermediate terms in the equation when \( h(x) \) is reduced. This implies that any other polynomial
with degree $< 4$ say a cubic or even quadratic polynomial should not have more than two intermediate terms after reduction. By definition 3.1, such polynomials belong to J-sac family and thus all proper polynomials of degree $\leq 4$ belong to J-sac family.

**Proposition 3.3.**

All proper polynomial equations of degree $\geq 5$ belong to B-sac family.

**Proof.**

Let $\delta(x) = x^5 + ax + b = 0$ where the degree of polynomial $k = 5$ with one intermediate term. It implies $\delta(x)$ is proper and thus $\delta(x) \in F$. Since $\delta_x(t) = t^5 + at^4 + bt^3 + ct^2 + e = 0$ has three intermediate terms in the equation, that are $at^4$, $bt^3$ and $ct^2$, it implies that any other polynomial with degree $> 5$ say a degree six polynomial (sextic polynomial) should have more than three intermediate terms after reduction. By definition 3.1, such polynomials belong to B-sac family and thus all proper polynomials of degree $\geq 5$ belong to B-sac family.

**Definition 3.4.**

Vic-emmense equations are simultaneous equations which involve algebraically closed polynomial equations. Two polynomial equations $f(x) = 0$ and $g(x) = 0$ are algebraically closed if there exists at least a root $\mu \in f(x) = 0$ and $g(x) = 0$. If such a condition is satisfied, then algebraically closed $f(x) = 0$ and $g(x) = 0$ are vic-emmense equations.

**Proposition 3.5.**

If $\mu \in f(x) = 0$ and $g(x) = 0$ conversely $f(x) = 0$ and $g(x) = 0$ can be solved vic-emmense to get the root $\mu$.

**Proof.**

In particular, let’s consider algebraically closed polynomial equations $f(x) = 2x^2 + x - 10 = 0$ and $g(x) = 5x^3 - 3x^2 - 5x - 18 = 0$ so that $x = 2 \in f(x) = 0$ and $g(x) = 0$.

Recall the two equations,

(3.5.1) $2x^2 + x - 10 = 0$

(3.5.2) $5x^3 - 3x^2 - 5x - 18 = 0$

Solving (3.5.1) and (3.5.2) vic-emmense, we have

(3.5.3) $91x - 182 = 0$ which implies,

(3.5.4) $x = 2$

Therefore the root $x = 2$ belongs to $g(x)$ and $f(x)$.

**Definition 3.6.**

Crack is a deduced equation from a proper polynomial equation and its reanegbèd equation which helps in reducing or rewriting proper polynomial equation.

**Proposition 3.7.**

The equation $\sigma = \frac{\Delta(t)}{\nabla(t)}$, $\nabla(t) \neq 0$ becomes a crack if and only if $\sigma$ are roots of a proper polynomial equation $f_{++}(x) = 0$ with corresponding roots of negation satisfying its reanegbèd equation.

**Proof.**

From definition 2.3, it shows that with reanegbèd equations, there is a relationship between a root and its negation ($t$). Solving the reanegbèd equation and its polynomial equation vic-emmense by proposition 3.5 can make the root a subject of its negation, that is $\sigma = \frac{\Delta(t)}{\nabla(t)}$ for each $\sigma$ having its $t$ for which $\nabla(t) \neq 0$.

**Corollary 3.8.** If $h(x)$ is the divisor of a dividend polynomial $f(x)$, then

(a) the degree of $h(x) \leq$ the degree of $f(x)$ and

(b) that the dividend polynomial has a coefficient of the leading dipole $\geq$ that of the divisor

(c) otherwise $h(x)$ is a dividend polynomial and $f(x)$ is the divisor.

**Proof.** Refer to division algorithm theorem.

**Characteristics of a Remainder:** Let $x - a$ be the divisor for a dividend polynomial $x^2 + bx + c$. By the remainder theorem, we have $x^2 + bx + c = (x - a) \cdot (x + a + b) + r(x)$, where $(x + a + b)$ is the quotient and $r(x) = a^2 + ab + c$ is the remainder.
If we change the divisor to a quadratic \( x^2 + px + q \) and the dividend polynomial to \( x^3 + ax^2 + bx + c \), then \( x^3 + ax^2 + bx + c = (x^2 + px + q) \cdot (x + a - p) + r(x) \) where again \( (x + a - p) \) will be the quotient and \( r(x) = (p^2 - ap - q + b)x + c + p - aq \) the remainder. It can be seen that, the remainder when the divisor is quadratic has a term in \( x \) while that of the divisor with linear \( x - a \) has no term in \( x \).

Again, let's take a cubic divisor \( x^3 + px^2 + qx + r \) and a dividend polynomial \( ax^3 + bx^2 + cx + d \), then \( ax^3 + bx^2 + cx + d = (x^3 + px^2 + qx + r) \cdot a + r(x) \) where \( a \) is the quotient and \( r(x) = (b - ap)x^2 + (c - aq)x + d - ar \) is the remainder. These follow immediately that,

**Corollary 3.10.** If a divisor has a degree \( k \), then the remainder \( r(x) \) forms a polynomial \( r(x) = a_0x^{k-1} + a_1x^{k-2} + \cdots + a_kx^0 \) \( \) where \( a_0, a_1, \ldots, a_{k-1}, a_k \) are called sub-expressions of the remainder. The sub-expressions becomes sub-equations if and only if the divisor is a factor of its dividend polynomial such that \( a_0 = 0, a_1 = 0, a_2 = 0, \ldots, a_{k-1} = 0, a_k = 0 \).

**Example:** Let \( x^3 + px^2 + q \) be the divisor of \( x^4 + 12x^3 + 35x^2 - 18x - 126 \). It follows from the remainder theorem that \( x^4 + 12x^3 + 35x^2 - 18x - 126 = (x + 12 - a)(x^3 + px^2 + q) + r(x) \). Since the divisor has degree 3, it implies from corollary 3.10, the remainder must form a polynomial in \( x \) of degree 2 such that \( r(x) = (35 + p^2 - 12p)x^2 + (-18 - q)x - 126 + pq - 12q \). The coefficients of \( r(x) \) which are called sub-expressions are \( (35 + p^2 - 12p), (-18 - q) \) and \(-126 + pq - 12q\). The sub-expressions become sub-equations that are \( 35 + p^2 - 12p = 0 \), \( -18 - q = 0 \) and \(-126 + pq - 12q = 0\). It is known that the divisor \( x^3 + px^2 + q \) is a factor of \( x^4 + 12x^3 + 35x^2 - 18x - 126 \). In a case where the divisor is a factor of the dividend polynomial, then from the sub-equations must be:

(i) \( 35 + p^2 - 12p = 0 \)
(ii) \(-18 - q = 0 \)
(iii) \(-126 + pq - 12q = 0\)

and from (ii) \( q = -18 \) and substituting it into (iii) will give
\(-18p + 90 = 0 \) or \( p = 5 \).

**Characteristics of vic-emmeuous polynomials:**
Two polynomial functions are vic-emmeuous, if there exist at least a zero in the two polynomials. Let us consider the vic-emmeuous functions
\( f(x) = 5x^2 - x - 18 \)
\( g(x) = 2x^2 + x - 10 \)
If we are to divide one function by the other, then the dividend function is \( f(x) \) and \( g(x) \) the divisor (refer to corollary 3.8).

By the remainder theorem, \( 5x^2 - x - 18 = \frac{5}{2} \).
\( 2x^2 + x - 10 + \left( \frac{7x}{2} + 7 \right) \) where \( \frac{5}{2} \) is the quotient and \( -\frac{7x}{2} + 7 \) the remainder. Since the two functions are vic-emmeuous, the remainder is referred to as an algebraic residue. Algebraic residue \( -\frac{7x}{2} + 7 \) when equated to zero will give a zero of \( f(x) \) and \( g(x) \) simultaneously at \( x = 2 \). Similarly for when the polynomial functions are equations, then simultaneously a root of \( f(x) \) and \( g(x) \) will also be \( x = 2 \).

If also \( x^4 + 10x^3 + 35x^2 + 50x + 24 \) and \( x^3 + 8x^2 + 17x + 10 \) are vic-emmeuous polynomials, then \( x^4 + 10x^3 + 35x^2 + 50x + 24 = (x + 2) \cdot (x^3 + 8x^2 + 17x + 10) + (2x^2 + 6x + 4) \) the remainder. The algebraic residue \( 2x^2 + 6x + 4 \) gives zeros of \( x^4 + 10x^3 + 35x^2 + 50x + 24 \) and \( x^3 + 8x^2 + 17x + 10 \) at \( x = -1 \) and \(-2 \).

An algebraic residue may form a polynomial of higher degree say degree 28. If that happens, we divide any of the vic-emmeuous polynomials by the algebraic residue and the process is repeated until we finally get an algebraic residue repeating itself. In that case, the algebraic residue repeating itself becomes the real and that of degree 28 becomes the virtual.

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Proposition 3.12. If $x^2 + qx + \beta$ is a factor of $x^4 + px^3 + kx^2 + rx + t$ and $x^4 + px^3 + kx^2 + rx + t$ is also a factor of $x^5 + ax^3 + bx^2 + cx + d$, then $x^2 + qx + \beta$ is also a factor of $x^5 + ax^3 + bx^2 + cx + d$ so that,

(3.12.0) \[ \Phi(q) = 0 \]

(3.12.1) \[ \zeta(q) = 0 \]

For which (3.12.0) and (3.12.1) are solved vic-emmeously to arrive at $q(\theta) = 0$ where degree of $q(\theta) < 5$.

Proof. Let $Z(x) = x^2 + qx + \beta$, $H(x) = x^4 + px^3 + kx^2 + rx + t$ and $Y(x) = x^5 + ax^3 + bx^2 + cx + d$. If $Z(x)|H(x)$ and $H(x)|Y(x)$ so must $Z(x)|Y(x)$ which will lead to $d + cp + bp^2 + ap^3 + p^5 = 0$ \( (\Delta) \)

\[
\begin{align*}
& b^2 + ap^4 + p^6 - 2p^5q - a^2q^2 + 4cq^2 - 2aq^4 - \frac{q^6}{p^3} + 2b - aq + 5q^3 + p^2(c - 3bq - aq^2 - 5q^4) + p(ab + a^2q - 4cq + 3bq^2 + 4aq^3 + 3q^5) = 0 \\
& \quad (V)
\end{align*}
\]

\[
\begin{align*}
& b^2c - abd + d^2 = (b^2 - a^2d + 4cd)q + \frac{(ab^2 - a^2c + 4c^2 - 7bd)q^2 + (a^2b - 4bc - 4ad)q^3 + (-a^3 - b^2 + 2ac)q^4 + (2ab - 11d)q^5 + (-3a^2 + 3c)q^6 + bq^7 - 3aq^8 - q^{10}}{2a_1} = 0 \\
& \quad (*)
\end{align*}
\]

where $p = \frac{-b_1 + \sqrt{b_1^2 - 4a_1c_1}}{2a_1}$ or $\frac{-b_1 - \sqrt{b_1^2 - 4a_1c_1}}{2a_1}$

and $a_1, c_1, b_1 \in q$

Solving (*) and (\(\Delta\)) vic-emmeously with one of $p$ substituted into (\(\Delta\)) will give

(3.12.0) \[ \Phi(q) = 0 \]

(3.12.1) \[ \zeta(q) = 0 \]

For which (3.12.0) and (3.12.1) are solved vic-emmeously to arrive at $q(\theta) = 0$ where degree of $q(\theta) = 4$. ■

Remark: Proposition 3.12 uses the transitive property of divisibility which states for example that if 3 divides 6 and 6 divides 12, then 3 must divide 12.

Note that our ultimate goal is to find for either $p$ or $q$. Since we can get four values of $q$ from solving $q(\theta) = 0$, it implies $x^4 + px^3 + kx^2 + rx + t$ or $x^2 + qx + \beta$ can be solved and thereby solving for $x^5 + ax^3 + bx^2 + cx + d$.

The E-Theory: This theory explains why degree two to four polynomials behave differently from degree five and above polynomials. Two equations are solved vic-emmeously if it is known that the equations are algebraically closed (at least a root belong to the equations). We also say basically that, a symmetric function of roots $\alpha$ and $\beta$ is one in which, if $\alpha$ and $\beta$ are interchanged, the function is the same or is multiplied by $-1$.

E-Theory states that, generally, all factors of a reduced cubic or quartic polynomial generate symmetric sub-equations in one variable to each other while quintic and above polynomials do not.

Investigating E-Theory: We will show that generally all reduced cubic and quartic polynomials have factors generating symmetric sub-equations in one variable to each other. We should take note that; the transitive property of divisibility is applied throughout in the investigation for identifying factors of a polynomial.

a) Let us consider the general quartic polynomial $x_1^4 + ax_1^3 + bx_1^2 + cx_1 + d$. Eliminating the cubic term in $x_1^4 + ax_1^3 + bx_1^2 + cx_1 + d$ will give a reduced polynomial $x^4 + b_1x^2 + c_1x + d_1$. Let $(r + kx + px^2 + x^3)$ and $(x^2 + qx + \beta)$ be the factors of $x^4 + b_1x^2 + c_1x + d_1$. It implies the factor $(r + kx + px^2 + x^3)$ will produce sub-equation of $4b_1^2q^2 - 4a_1^2q^4 + 16c_1q^4 - 8a_1q^6 - 4q^8 = 0$ while the factor $(x^2 + qx + \beta)$ will also produce sub-equation of $b_1^2 - a_1^2q^2 + 4c_1q^2 - 2a_1q^4 - q^6 = 0$. Both sub-equations are symmetric.

b) Let us also consider the general cubic polynomial $x^3 + ax^2 + bx + c$. Eliminating
the quadratic term in \( x^3 + ax^2 + bx + c \) will give a reduced polynomial \( x_1^3 + a_1x_1 + b_1 \). Let \( (x_1^2 + c_1x_1 + d_1) \) and \( (x_1 + k) \) be the factors of \( x_1^3 + b_1x_1 + c_1 \). It implies the factor \( (x_1^2 + c_1x_1 + d_1) \) will produce sub-equation of \( b_1 + a_1c_1 + c_1^3 = 0 \) while the factor \( (x_1 + k) \) will also produce sub-equation of \( b_1 + a_1c_1 + c_1^3 = 0 \). Both sub-equations are symmetric.

c) For quintic polynomials and above, Proposition 3.12 makes it clear that, the factors do not generate symmetric sub-equations generally when the polynomial is reduced.

**Proposition 3.15.** If the standard quintic equation is of the form \( h(t) = t^5 + at^3 + bt^2 + ct + d = 0 \), then \( h(t) = 0 \in B\text{-sac family} \) and also solvable algebraically in terms of radicals.

**Proof.** By proposition 3.3, \( h(t) = 0 \in B\text{-sac family} \) and let \( \varphi(t) \) be a quadratic factor and \( \omega(t) \) be a cubic factor. In order to satisfy \( h(t) = 0 \), then by proposition 3.12, two sub-equations in \( q \) must be

\[
\begin{align}
\nabla(q) &= 0 \\
\Delta(q) &= 0
\end{align}
\]

For which (3.15.1) and (3.15.2) are solved vic-enemously to arrive at \( q(t) = \) where degree of \( q(t) = 4 \). Solving for \( q(t) = 0 \) will eventually lead to getting all the values for \( p \) in the equation \( d + cp + bp^2 + ap^3 + p^5 = 0 \)

Since \( p \) is known it implies \( \varphi(t) \) and \( \omega(t) \) are satisfied and thus solving \( h(t) = 0 \).

**Proposition 3.16.** All \( f(x) = 0 \in B\text{-sac family} \) can be solved algebraically in terms of radicals.

**Proof.** In particular, let \( f(x) = x^5 + a_1x^4 + b_1x^3 + c_1x^2 + d_1x + e_1 \) so that \( f(x) = 0 \in B\text{-sac} \) by proposition 3.3 and be the general form of quintic equation. Thus the reanegbèd equation of \( f(x) = 0 \) is \( g(x, t) = 120x + 24a_1 + t = 0 \) so that the quartic term will be eliminated in the reduced equation. Recall

\[
\begin{align}
\text{(3.16.1)} \quad g(x, t) &= 0 \\
\text{(3.16.2)} \quad x &= h(t)
\end{align}
\]

Substituting (3.16.2) into \( f(x) = 0 \), we have

\[
\begin{align}
f[h(t)] &= 0 \quad \text{and letting } f[h(t)] = \tau(t), \text{ it implies } \\
\tau(t) &= t^5 + bt^3 + ct^2 + dt + e = 0 \text{. Let }
\end{align}
\]

\[
\begin{align}
\text{(3.16.3)} \quad \tau(t) &= 0 \\
\text{(3.16.4)} \quad \nabla(q) &= 0 \\
\text{(3.16.5)} \quad \Delta(q) &= 0
\end{align}
\]

For which (3.16.5) and (3.16.4) are solved vic-enemously to arrive at \( q(t) = \) where degree of \( q(t) = 4 \).

Solving for \( q(t) = 0 \) will eventually lead to getting all the values for \( p \) in the equation \( e + dp + cp^2 + bp^3 + p^5 = 0 \)

Since \( p \) is known it implies \( \varphi(t) \) and \( \omega(t) \) are satisfied and thus solving \( \tau(t) = 0 \). Since the five values of \( t \) are known, we can get the five roots of \( x \) from \( x = h(t) \). Similar steps can be used to solve sextic equations and higher. ■

**Proposition 3.17.** \( x^5 + ax + b = 0 \in B\text{-sac family} \) and can be solved algebraically in terms of radicals.

**Proof.** Let \( f(x) = x^5 + a_1x + b_1 \), so that \( f(x) = 0 \) and thus the reanegbèd equation of \( f(x) = 0 \) will be \( g(x, t_1) = 5x^4 + a_1 + t_1 = 0 \), so that after reduction, \( f(x) = 0 \in B\text{-sac} \).

Recall
(3.17.1) \[ g(x, t_1) = 0 \quad \text{and} \]
(3.17.2) \[ f(x) = 0 \]

By proposition 3.5, (3.17.2) and (3.17.1) can be solved vic-emmeously to get

(3.17.3) \[ x = h(t_1) \]

Substituting (3.17.3) into \( f(x) = 0 \), we have

\[ f[h(t_1)] = 0 \quad \text{and letting} \quad f[h(t_1)] = \tau(t_1), \]
implies \( \tau(t_1) = t_1^5 + \alpha t_1^4 + \beta t_1^3 + \gamma t_1^2 + \delta = 0 \)

(3.17.4) \[ \tau(t_1) = 0 \]

Reduce (3.17.4) to the standard form equation by choosing \( g(t_1, t) = 120t_1 + 24\alpha + t = 0 \) as the reanegbèd equation so that the quartic term will be eliminated in the reduced equation.

Recall

(3.17.5) \[ g(t_1, t) = 0 \quad \text{and it implies} \]
(3.17.6) \[ t_1 = k(t) \]

Substituting (3.17.6) into (3.17.4), we have

\[ \tau[k(t)] = 0 \quad \text{and letting} \quad \tau[k(t)] = M(t), \]
implies \( M(t) = t^5 + bt^3 + ct^2 + dt + e = 0 \)

Let

(3.17.7) \[ M(t) = 0 \]

so that \( \rho(t) \) will be a quadratic factor and \( \sigma(t) \)
will be a cubic factor. In order to satisfy \( M(t) = 0 \),
then by proposition 3.12, two sub-equations in \( q \)
must be

(3.17.8) \[ \nabla(q) = 0 \quad \text{and} \]
(3.17.9) \[ \Delta(q) = 0 \]

For which (3.17.9) and (3.17.8) are solved vic-emmeously to arrive at \( q(\theta) = 0 \) where degree of \( q(\theta) = 4 \). Solving for \( q(\theta) = 0 \) will eventually lead to getting all the values for \( p \) in the equation \( e + dp + cp^2 + bp^3 + p^5 = 0 \). Since \( p \) is known it implies \( \rho(t) \) and \( \sigma(t) \) are satisfied and thus solving \( M(t) = 0 \). Since the five values of \( t \) are known, we can also get the five values of \( t_1 \) from \( t_1 = k(t) \) and finally \( x \) from \( x = h(t_1) \).

CONCLUSION

Proving proposition 3.12 has been the greatest challenge in the world of mathematics for many years now. The proof indicates that for such a solution to exist; then the polynomial equations

should be proper polynomial belong to B-sac family. E-Theory explains why polynomials belonging to J-sac family behave differently from that of B-sac family in terms of solution.

Solutions of polynomial equations \( f(x) = 0 \in J-

sac family are well-known, as Gerolamo Cardano (1545), Lodovico Ferrari (1522 –1565) and others rigorously demonstrated.

Solutions of improper polynomial equations are well known to go by and these conclude therefore that polynomial equations of degree \( n \geq 1 \) where \( n \) is a positive integer are solvable algebraically in terms of finite number of additions, subtractions, multiplications, divisions, and root extractions.

REFERENCES


